Pigeonhole Principle and its Applications

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Outline

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- Discussion and Conclusion

Basic Version

Theorem: (PHP)
Let n, k be positive integers with $n > k$. If n objects are placed into k boxes, then there is a box that has at least 2 objects in it.

Proof:
We will implement this proof by **contradiction**. Suppose that this statement is false. Then each box has either 0 or 1 object in it. Let m be the number of boxes that have 1 object in it. Then there are m objects total and hence $n = m$. However $m \leq k$ since there are k boxes, and this is equivalent to $n \leq k$. But obviously this contradicts to our assumption that $n > k$.

How can this seemingly "trivial" principle be implemented in simple questions?

1) If we have 4 flagpoles and we put up 5 flags, then there is some flagpole that has at least 2 flags on it.
2) At least 2 of the people in our class (12 students + 1 instructor) were born in the same month.
3) Pick 5 different integers between 1 and 8. Then there must be a pair of them that adds up to 9.
4) In any finite graph, there are two vertices of equal degree.

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General Version

Theorem: (General PHP)
Let n, k, r be positive integers and suppose that $n > rk$. If n objects are placed into k boxes, then there is a box that contains at least $r + 1$ objects in it.
Clearly, if $r = 1$, it will be the basic version of PHP.

General Version

Proof:

We can again implement this proof by **contradiction**. Suppose the statement is false and label the boxes from 1 up to k. Let b_i be the number of objects in box number i. Then $b_i \leq r$ since we have assumed that the statement is false. Furthermore, we have $n = b_1 + b_2 + \cdots + b_k \leq r + r + \cdots + r = rk$. But this contradicts the assumption that $n > rk$.

Example:

If we have 4 flagpoles and 9 flags distributed to them, then some flagpole must have at least 3 flags on it.

Infinite Case - Ramsey Theorem

Claim:
Given one infinite sequence of *distinct* real numbers $x_1, x_2, \cdots$, by **Ramsey Theorem**, there is an infinite monochromatic subset, here "monochromatic" means that it contains a monotonically increasing or decreasing infinite subsequence. This can be achieved by 2-colouring method, with the pair $\{i, j\}$ of positive integers (where $i < j$) red if $x_i < x_j$, green if $x_i > x_j$.

Motivation:
But further we are eager to find a finitary result that makes it more precise in a finite sequence.

Erdős-Szekeres Theorem

Theorem:

Given m, n, any sequence of *distinct* real numbers with length of at least $mn+1$ $\{x_1, x_2, \cdots, x_{mn+1}\}$ contains a monotonically increasing subsequence of length $m+1$ or a monotonically decreasing subsequence of length $n+1$.

Erdős-Szekeres Theorem - Proof

Proof:
For $i = 1, 2, \cdots, mn + 1$, let s_i denote the length of a longest **increasing** subsequence that ends at x_i and t_i denote the length of a longest **decreasing** subsequence that ends at x_i.
Suppose the sequence has increasing subsequences of length at most m and decreasing subsequences of lengths at most n.
That is, for all $i = 1, 2, \cdots, mn + 1$, we have

$$1 \leq s_i \leq m, \quad 1 \leq t_i \leq n$$

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Erdős-Szekeres Theorem - Proof (Cont.)

So there are at most mn distinct possible tuples (s_i, t_i),
Since there are $mn + 1$ tuples (s_i, t_i) corresponding to the term x_i (They are bijective!), by **Pigeonhole Principle**, there exist some $1 \leq j < k \leq mn + 1$ such that

$$(s_j, t_j) = (s_k, t_k)$$

Since they are distinct, now either $x_j > x_k$ or $x_j < x_k$.

Erdős-Szekeres Theorem - Proof (Cont.)

If $x_j > x_k$, then appending x_k to a longest decreasing subsequence ending at x_j would give a decreasing subsequence of length $t_j + 1$ ending at x_k. By definition of t_k, we must have $t_k \geq t_j + 1$, which contradicts to $t_j = t_k$.

If $x_j < x_k$, similarly, then appending x_k to a longest increasing subsequence ending at x_j would give an increasing subsequence of length $s_j + 1$ ending at x_k. By definition of s_k, we must have $s_k \geq s_j + 1$, which contradicts to $s_j = s_k$.

Hence, no such j and k exist, and thus this sequence contains a monotonically increasing subsequence of length $m + 1$ or a monotonically decreasing subsequence of length $n + 1$. $\square$

Dirichlet's Approximation Theorem

Theorem:

For any *irrational* number α and an integer N, there exist integers p and q such that, for $1 \leq q \leq N$,

$$|\alpha - \frac{p}{q}| < \frac{1}{q^2}$$

Dirichlet's Approximation Theorem - Proof

Proof:

Let α be an irrational number and N be an integer. For every $k = 0, 1, \cdots, N$, we can write $k\alpha = m_k + x_k$ such that m_k is an integer and $0 \leq x_k < 1$. Then we can divide the interval $[0, 1)$ into n smaller intervals of measure $\frac{1}{N}$. Now, we have $N + 1$ numbers $x_0, x_1, ..., x_N$ and n intervals. Therefore, by the **Pigeonhole Principle**, at least two of them are in the same interval. We can call those x_i, x_j such that $i < j$. Now we have

$$|x_j - x_i| = |j\alpha - m_j - (i\alpha - m_i)| = |(j - i)\alpha - (m_j - m_i)| < \frac{1}{N}$$

Dirichlet's Approximation Theorem - Proof (Cont.)

Proof:

Then we can divide both sides by $j - i$ to obtain

$$|\alpha - \frac{m_j - m_i}{j - i}| < \frac{1}{(j-i)N} \leq \frac{1}{(j-i)^2}$$

Here $m_j - m_i \in \mathbb{N}, j - i \in \mathbb{N}$, and $j - i \leq N$.

Thus, we finish the proof. $\square$

Common in High-Level Math Competition

IMO 1972/1

Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

Manhattan Mathematical Olympiad 2003/4

Prove that from any set of one hundred whole numbers, one can choose either one number which is divisible by 100, or several numbers whose sum is divisible by 100.

Manhattan Mathematical Olympiad 2005/1

Prove that having 100 whole numbers, one can choose 15 of them so that the difference of any two is divisible by 7.

PHP Process

Characteristic of problems solved by PHP: *Quick and beautiful*.
It usually contains a process with three parts:

- Recognize that the problem requires the Pigeonhole Principle;
- Figure out what the pigeons and what the pigeonholes might be;
- Keep on deconstructing the problems and move forward!

The END

Thanks For Listening!